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SOLUTION OF THREE-DIMENSIONAL PROBLEMS OF THE THEORY OF ELASTICITY FOR SOLIDS OF REVOLUTION BY MEANS OF ANALYTICAL FUNCTIONS

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Abstract—By means of superposition relations are established between space states on the one hand, and elementary auxiliary states, (plane states and states corresponding to the deplanation of plane sections of cylindrical bodies), on the other hand. In the particular case it is the relations between axisymmetric space and plane states.

The derived relations are employed for the solution of spatial problems by means of the apparatus of the theory of analytical functions of a complex variable, whose application to plane problems is widely known [1].

1. RELATIONS BETWEEN SPACE STATES AND ELEMENTARY STATES CONNECTED WITH THEM

1.1 *Space state*

BE IT supposed that the elastic cylinder with its generators parallel to the axis y, and with its cross-section symmetrical in relation to the axis z, is in a state formed by the superposition of the plane strain ($\tau_{yz} = \tau_{xy} = v = 0$) and the state corresponding to the deplanation of the cross-sections of the cylinder in the direction of the axis y

$$(\sigma_y = \sigma_x = \sigma_z = \tau_{xz} = w = u = 0), \text{ Fig. 1.}$$

The states of stress and strain of the cylinder we shall consider as being dependent on the parameter λ, whose value can be changed. The cylinder may be isotropic or transversely-isotropic with an axis of elastic symmetry z, homogeneous or heterogeneous with elastic parameters dependent only on the coordinate z.

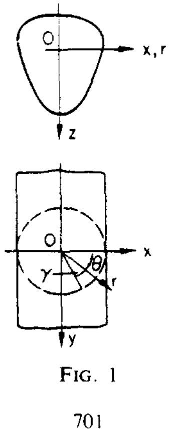

FIG. 1

By rotating the contour of the cross-section of the cylinder in relation to the axis z, let us excise from the cylinder a solid of revolution. It should be noted that such an operation is possible not for just any form of the cross-section. Therefore, let us temporarily introduce the restrictions making this operation possible, assuming that the function $x(z) = r(z)$ for half the contour, lying to one side of the axis z, is a single-valued function.

The surface of the excised solid of revolution will fall under the action of certain forces. If we were now to displace these forces along the surface of the solid by revolving around the axis z, the internal stress and displacement will follow the forces without changing in value, provided the parameter λ does not change.

In the excised solid let us pass from the components of stress and displacement in the Cartesian coordinates x, y, z to the components in the cylindrical coordinates r, θ, z and superimpose these components during their revolution in relation to the solid around the axis z, with simultaneous variations of parameter λ. We shall consider this parameter to be equal to the angle of rotation γ. Rotating through the angle 2π we shall get a three-dimensional state of the solid of revolution whose components are defined by integration [9, 12, 13]

$$\sigma_z = \int_0^{2\pi} \sigma_{z\lambda}\, d\gamma,$$

$$\sigma_r = \int_0^{2\pi} [\sigma_{x\lambda} \cos^2(\gamma+\theta)+\sigma_{y\lambda} \sin^2(\gamma+\theta)+\tau_{xy\lambda} \sin 2(\gamma+\theta)]\, d\gamma, \tag{1.1}$$

$$\sigma_\theta = \int_0^{2\pi} [\sigma_{x\lambda} \sin^2(\gamma+\theta)+\sigma_{y\lambda} \cos^2(\gamma+\theta)-\tau_{xy\lambda} \sin 2(\gamma+\theta)]\, d\gamma;$$

$$\tau_{r\theta} = \int_0^{2\pi} [-\tfrac{1}{2}(\sigma_{x\lambda}+\sigma_{y\lambda}) \sin 2(\gamma+\theta)+\tau_{xy\lambda} \cos 2(\gamma+\theta)]\, d\gamma,$$

$$\tau_{zr} = \int_0^{2\pi} [\tau_{zx\lambda} \cos(\gamma+\theta)+\tau_{zy\lambda} \sin(\gamma+\theta)]\, d\gamma,$$

$$\tau_{z\theta} = \int_0^{2\pi} [-\tau_{zx\lambda} \sin(\gamma+\theta)+\tau_{zy\lambda} \cos(\gamma+\theta)]\, d\gamma; \tag{1.2}$$

$$w = \int_0^{2\pi} w_\lambda\, d\gamma,$$

$$u = \int_0^{2\pi} [u_\lambda \cos(\gamma+\theta)+v_\lambda \sin(\gamma+\theta)]\, d\gamma,$$

$$v = \int_0^{2\pi} [-u_\lambda \sin(\gamma+\theta)+v_\lambda \cos(\gamma+\theta)]\, d\gamma.$$

Here $\sigma_z(z, r, \theta)$, σ_r, $\tau_{r\theta}$ ... $w(r, \theta, z)$, u, v are stresses and displacements (axial, radial, tangential) of the three-dimensional state; $w_\lambda(x, z, \gamma)$, u_λ, v_λ—will be the displacements of the auxiliary states in the direction of axes z, x, y; the stresses $\sigma_{x\lambda}(x, z, \gamma)$, $\sigma_{y\lambda}$, $\sigma_{z\lambda}$, $\tau_{xz\lambda}$ correspond to plane strain, $\tau_{xy\lambda}(x, z, \gamma)$ and $\tau_{yz\lambda}$ correspond to deplanation, $\sigma_{y\lambda} = \nu(\sigma_{z\lambda}+\sigma_{x\lambda})$ for the isotropic and $\sigma_{y\lambda} = \nu_{xy}\sigma_{x\lambda}+\nu_{xz}(E_x/E_z)\sigma_{z\lambda}$ for the transversely-isotropic body, ν, ν_{xy}, ν_{xz} are Poisson's ratio; E_x, E_z are moduli of elasticity; $x = r\cos(\gamma+\theta)$.

Let us now establish the relations between the three-dimensional and auxiliary states by means of another superposition. Let us consider a space with an axisymmetric cavity in a general state of stress (Fig. 2). By displacing the contour of the meridional section of the cavity along the axis η from $\eta = -\infty$ to $\eta = \infty$ let us excise a cylindrical cavity* (axis η lies in the plane xy at the angle γ to axis y). Be it noted that such an operation is possible not just with any form of the contour of the meridional section of the cavity and therefore we shall temporarily introduce the same restrictions as for the contour of the cross-section of the cylinder. For a space with such an excised cylindrical cavity let us pass from stress and displacement components of three-dimensional states in cylindrical co-ordinates r, θ, z to components in Cartesian co-ordinates η, ξ, z and superimpose the stress and displacement components by moving them in relation to space along the axis η from $\eta = -\infty$ to $\eta = \infty$. As a result of such superposition we shall derive a state of stress and strain of space formed by the superposition of two states—the plane strain

$$(\tau_{\eta z} = \tau_{\xi\eta} = v_\eta = 0)$$

and the state of deplanation ($\sigma_\xi = \sigma_\eta = \sigma_z = \tau_{\xi z} = w = u_\xi = 0$). The components of this state will depend on the parameter γ and are defined by the expressions [9], [12], [13].

$$\begin{aligned}
\sigma_{z\|} &= \int_{-\infty}^{\infty} \sigma_z \, d\eta, \\
\sigma_{\xi\|} &= \int_{-\infty}^{\infty} [\sigma_r \cos^2(\theta-\gamma)+\sigma_\theta \sin^2(\theta-\gamma)-\tau_{r\theta} \sin 2(\theta-\gamma)] \, d\eta, \\
\sigma_{\eta\|} &= \int_{-\infty}^{\infty} [\sigma_r \sin^2(\theta-\gamma)+\sigma_\theta \cos^2(\theta-\gamma)+\tau_{r\theta} \sin 2(\theta-\gamma)] \, d\eta; \\
\tau_{\xi\eta\|} &= \int_{-\infty}^{\infty} [\tfrac{1}{2}(\sigma_r-\sigma_\theta) \sin 2(\theta-\gamma)+\tau_{r\theta} \cos 2(\theta-\gamma)] \, d\eta, \\
\tau_{z\xi\|} &= \int_{-\infty}^{\infty} [\tau_{zr} \cos(\theta-\gamma)-\tau_{z\theta} \sin(\theta-\gamma)] \, d\eta, \\
\tau_{z\eta\|} &= \int_{-\infty}^{\infty} [\tau_{zr} \sin(\theta-\gamma)+\tau_{z\theta} \cos(\theta-\gamma)] \, d\eta;
\end{aligned} \tag{1.3}$$

$$\begin{aligned}
w_{\|} &= \int_{-\infty}^{\infty} w \, d\eta, \\
u_{\xi\|} &= \int_{-\infty}^{\infty} [u \cos(\theta-\gamma)-v \sin(\theta-\gamma)] \, d\eta, \\
v_{\eta\|} &= \int_{-\infty}^{\infty} [u \sin(\theta-\gamma)+v \cos(\theta-\gamma)] \, d\eta.
\end{aligned} \tag{1.4}$$

Here $\sigma_{z\|}(\xi, z, \gamma)$, $\sigma_{\xi\|}$, $\sigma_{\eta\|}$, $\tau_{\xi z\|}$, $w_{\|}$, $u_{\xi\|}$ correspond to plane strain, and $\tau_{\xi\eta\|}(\xi, z, \gamma)$, $\tau_{\eta z\|}$, $v_{\eta\|}$ signifies deplanation. For an isotropic body $\sigma_{\eta\|} = \nu(\sigma_{z\|}+\sigma_{\xi\|})$, for a transversely-isotropic body $\sigma_{\eta\|} = \nu_{\xi\eta}\sigma_{\xi\|}+\nu_{\xi z}(E_x/E_z)\sigma_{z\|}$.

* This cavity is shown by a straight dotted line in the bottom view of Fig. 2.

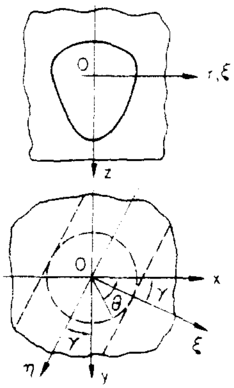

FIG. 2

1.2 *The axisymmetric state*

Let us derive the relations between the components of a three-dimensional axisymmetric and auxiliary plane state as a particular case from the formulae (1.1), (1.2), or by means of superposition analogous to the one described above. In this case the problem is simplified — the components of the state of deplanation of a cylinder should be taken as equal to zero, whereas the parameter λ should not be introduced into the expressions of the plane state components (that is, we shall not alter the plane state during the rotation of the cylinder). It will be assumed that the plane state of the cylinder is symmetrical in relation to plane yz and we shall derive an axisymmetric state by means of superposition while rotating the plane state through the angle π (and not 2π). Thus, we shall arrive at [2–5, 13]

$$\sigma_r = \int_0^{\pi} (\sigma_{x\parallel} \cos^2\theta + \sigma_{y\parallel} \sin^2\theta)\,d\theta, \qquad \sigma_z = \int_0^{\pi} \sigma_{z\parallel}\,d\theta,$$
$$\sigma_\theta = \int_0^{\pi} (\sigma_{y\parallel} \cos^2\theta + \sigma_{x\parallel} \sin^2\theta)\,d\theta, \qquad \tau_{rz} = \int_0^{\pi} \tau_{xz\parallel} \cos\theta\,d\theta; \tag{1.5}$$

$$u = \int_0^{\pi} u_{\parallel} \cos\theta\,d\theta, \qquad w = \int_0^{\pi} w_{\parallel}\,d\theta. \tag{1.6}$$

Through the action of the plane system of body forces $X_{\parallel}(x, z)$ and $Z_{\parallel}(x, z)$ on the cylinder we shall derive in the solid of revolution an axisymmetric system of body forces [5] $R(r, z)$ and $Z(r, z)$

$$R = \int_0^{\pi} X_{\parallel} \cos\theta\,d\theta, \qquad Z = \int_0^{\pi} Z_{\parallel}\,d\theta. \tag{1.7}$$

We shall now obtain the relations between the axisymmetric and auxiliary plane states by means of superposition analogous to the one described above, [or directly from the formulae (1.3), (1.4)]. Inasmuch as in this case the angle γ does not affect the components of the plane state, obtained by superposition, we shall presume that $\gamma = 0$, that is, introduce integration with respect to y from $y = -\infty$ to $y = \infty$ (Fig. 3). Herein, the components representing deplanation, are converted to zero.

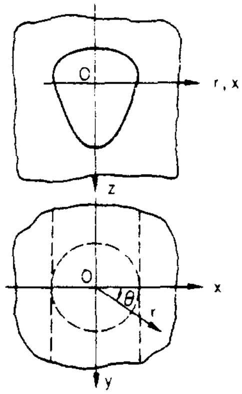

FIG. 3

As a result, we shall find [2–5, 13]

$$\sigma_{x\parallel} = \int_{-\infty}^{\infty} (\sigma_r \cos^2\theta + \sigma_\theta \sin^2\theta)\,\mathrm{d}y, \qquad \sigma_{z\parallel} = \int_{-\infty}^{\infty} \sigma_z\,\mathrm{d}y,$$
$$\sigma_{y\parallel} = \int_{-\infty}^{\infty} (\sigma_r \sin^2\theta + \sigma_\theta \cos^2\theta)\,\mathrm{d}y, \qquad \tau_{xz\parallel} = \int_{-\infty}^{\infty} \tau_{rz}\cos\theta\,\mathrm{d}y, \tag{1.8}$$

$$u_\parallel = \int_{-\infty}^{\infty} u\cos\theta\,\mathrm{d}y, \qquad w_\parallel = \int_{-\infty}^{\infty} w\,\mathrm{d}y. \tag{1.9}$$

When body forces are involved

$$X_\parallel = \int_{-\infty}^{\infty} X\cos\theta\,\mathrm{d}y, \qquad Z_\parallel = \int_{-\infty}^{\infty} Z\,\mathrm{d}y. \tag{1.10}$$

The derived relations between the plane and axisymmetric state components shall be regarded as integral equations with respect to the components of the axisymmetric state. Let us reduce these equations to the Abel-type equations and solve them. As a result we shall obtain

$$\sigma_r + \sigma_\theta = -\frac{1}{\pi}\int_r^\infty \frac{\partial(\sigma_{x\parallel} + \sigma_{y\parallel})}{\partial x}\frac{\mathrm{d}x}{\sqrt{(x^2-r^2)}},$$
$$\sigma_z = -\frac{1}{\pi}\int_r^\infty \frac{\partial\sigma_{z\parallel}}{\partial x}\frac{\mathrm{d}x}{\sqrt{(x^2-r^2)}},$$
$$\sigma_r - \sigma_\theta = -\frac{1}{\pi}\int_r^\infty \frac{\partial(\sigma_{x\parallel} - \sigma_{y\parallel})}{\partial x}\frac{2x^2-r^2}{r^2\sqrt{(x^2-r^2)}}\,\mathrm{d}x + \frac{c_1}{r^2}, \tag{1.11}$$
$$\tau_{rz} = -\frac{1}{\pi}\int_r^\infty \frac{\partial\tau_{xz\parallel}}{\partial x}\frac{x\,\mathrm{d}x}{r\sqrt{(x^2-r^2)}},$$
$$\left(c_1 = \frac{2}{\pi}\lim[x(\sigma_x - \sigma_y)] \quad \text{as } x \to \infty\right).$$

$$u = -\frac{1}{\pi}\int_r^{\infty} \frac{\partial u}{\partial x}\,\frac{x\,dx}{r\sqrt{(x^2-r^2)}}, \qquad w = -\frac{1}{\pi}\int_r^{\infty} \frac{\partial w}{\partial x}\,\frac{dx}{\sqrt{(x^2-r^2)}} \tag{1.12}$$

$$R = -\frac{1}{\pi}\int_r^{\infty} \frac{\partial R}{\partial x}\,\frac{x\,dx}{r\sqrt{(x^2-r^2)}}, \qquad Z = -\frac{1}{\pi}\int_r^{\infty} \frac{\partial Z}{\partial x}\,\frac{dx}{\sqrt{(x^2-r^2)}} \tag{1.13}$$

2. REPRESENTATIONS OF THREE-DIMENSIONAL STATE COMPONENTS OF A HOMOGENEOUS ISOTROPIC BODY IN TERMS OF ANALYTICAL FUNCTIONS OF A COMPLEX VARIABLE OR CAUCHY-TYPE INTEGRALS

2.1 *Space state*

In the relations (1.1), (1.2) let us pass from integration with respect to θ to integration with respect to x. By utilizing the Kolosov–Muskhelishvili formulae and the solution of the torsion problem, let us represent the elementary state components in terms of functions $\varphi(\zeta, \gamma)$, $\psi(\zeta, \gamma)$ and $\Phi(\zeta, \gamma)$ of a complex variable $\zeta = z + ix$, also dependent on the parameter γ. Let us presume that as related to γ these functions may be expanded in trigonometric series

$$\varphi(\zeta, \gamma) = \sum_{n=-\infty}^{\infty} \varphi_n(\zeta)\,e^{-in\gamma}, \quad \psi(\zeta, \gamma) = \sum_{n=-\infty}^{\infty} \psi_n(\zeta)\,e^{-in\gamma}, \quad \Phi(\zeta, \gamma) = i\sum_{n=-\infty}^{\infty} \Phi_n(\zeta)\,e^{-in\gamma} \tag{2.1}$$

where $\varphi_n(\zeta)$, $\psi_n(\zeta)$ and $\Phi_n(\zeta)$ are functions analytical in the region occupied by the meridional section of the body satisfying the conditions $\varphi_n(\zeta) = (-1)^n\overline{\varphi_{-n}(\bar{\zeta})}$, $\psi_n(\zeta) = (-1)^n\overline{\psi_{-n}(\bar{\zeta})}$, $\Phi_n(\zeta) = (-1)^n\overline{\Phi_{-n}(\bar{\zeta})}$.

By applying the properties of analytical functions, let us make some transformations and obtain [9, 12, 13]

$$\begin{aligned}
\sigma_z &= \sum_{n=-\infty}^{\infty} \frac{e^{in\theta}}{\pi i}\int_{\bar{t}}^{t} [2\varphi_n'(\zeta) - (2z-\zeta)\varphi_n''(\zeta) - \psi_n'(\zeta)] \\
&\quad \times T_n\left(\frac{\zeta - z}{ri}\right)\frac{d\zeta}{\sqrt{[(\zeta - t)(\zeta - \bar{t})]}}, \\
\sigma_r + \sigma_\theta &= \sum_{n=-\infty}^{\infty} \frac{e^{in\theta}}{\pi i}\int_{\bar{t}}^{t} [2(1+2\nu)\varphi_n'(\zeta) - (2z-\zeta)\varphi_n''(\zeta) + \psi_n'(\zeta)] \\
&\quad \times T_n\left(\frac{\zeta - z}{ri}\right)\frac{d\zeta}{\sqrt{[(\zeta - t)(\zeta - \bar{t})]}}, \\
\sigma_r - \sigma_\theta + 2i\tau_{r\theta} &= \sum_{n=-\infty}^{\infty} \frac{e^{in\theta}}{\pi i}\int_{\bar{t}}^{t} [2(1-2\nu)\varphi_n'(\zeta) + (2z-\zeta)\varphi_n''(\zeta) \\
&\quad + \psi_n'(\zeta) + i\Phi_n'(\zeta)]\,T_{n+2}\left(\frac{\zeta - z}{ri}\right)\frac{d\zeta}{\sqrt{[(\zeta - t)(\zeta - \bar{t})]}}
\end{aligned} \tag{2.2}$$

$$\tau_{zr}+i\tau_{z\theta} = -\sum_{n=-\infty}^{\infty}\frac{e^{in\theta}}{\pi}\int_{\bar{t}}^{t}\left[(2z-\zeta)\varphi_n''(\zeta)+\psi_n'(\zeta)+\frac{i}{2}\Phi_n'(\zeta)\right] \times T_{n+1}\left(\frac{\zeta-z}{ri}\right)\frac{d\zeta}{\sqrt{[(\zeta-t)(\zeta-\bar{t})]}};$$

$$w = \sum_{n=-\infty}^{\infty}\frac{e^{in\theta}}{2\mu\pi i}\int_{\bar{t}}^{t}[\varkappa\varphi_n(\zeta)-(2z-\zeta)\varphi_n'(\zeta)-\psi_n(\zeta)] \times T_n\left(\frac{\zeta-z}{ri}\right)\frac{d\zeta}{\sqrt{[(\zeta-t)(\zeta-\bar{t})]}}. \tag{2.3}$$

$$u+iv = -\sum_{n=-\infty}^{\infty}\frac{e^{in\theta}}{2\mu\pi}\int_{\bar{t}}^{t}[\varkappa\varphi_n(\zeta)+(2z-\zeta)\varphi_n'(\zeta)+\psi_n(\zeta) +i\Phi_n(\zeta)]T_{n+1}\left(\frac{\zeta-z}{ri}\right)\cdot\frac{d\zeta}{\sqrt{[(\zeta-t)(\zeta-\bar{t})]}}.$$

Here $t = z+ir$, $\varkappa = 3-4\nu$, μ will be Lamé's constant, $T_n[(\zeta-z)/ri]$ is Chebyshev's polynomial of the first type ($T_{-n} = T_n$). For definiteness let us select one of the branches of the root $\sqrt{[(\zeta-t)(\zeta-\bar{t})]}$.

Integration from $\bar{t}$ to t may be conducted along any line lying on the meridional section of the body. It follows from this that the restrictions superimposed in 1.1 on the contour of this section may be removed [7].

The analytical functions involved in (2.2), (2.3) may be represented by Cauchy-type integrals. By changing the order of integration and computing the internal integrals, it becomes possible to express the components of the three-dimensional state not in terms of analytical functions but by the density of the Cauchy-type integrals [11].

2.2 *The axisymmetric state*

The representations of the axisymmetric state components may be derived from the expressions (2.2), (2.3) or by the procedure analogous to the one described above. In the last case let us introduce into the relations (1.5), (1.6) the expressions of the plane state components in terms of two analytical functions $\varphi(\zeta)$ and $\psi(\zeta)$. We shall use for this purpose the Kolosov–Muskhelishvili formulae with terms introduced by Stevenson, that would account for body forces. Let us take into consideration that because of the symmetry of the plane state in relation to the axis z, functions φ and ψ possess the property of parity

$$\varphi(\zeta) = \overline{\varphi(\bar{\zeta})}, \qquad \psi(\zeta) = \overline{\psi(\bar{\zeta})}. \tag{2.4}$$

In the expressions (1.5), (1.6) let us pass from integration with respect to θ to integration with respect to x and after the transformation obtain [2–5, 13]

$$\sigma_z = \frac{1}{\pi i}\int_{\bar{t}}^{t}[2\varphi'(\zeta)-(2z-\zeta)\varphi''(\zeta)-\psi'(\zeta)]\frac{d\zeta}{\sqrt{[(\zeta-t)(\zeta-\bar{t})]}} + \left\{-\frac{1}{2(1-\nu)}U(r,z)+\frac{c}{2}\int_{L_0}\mathrm{Re}\, W^*(x,z)\frac{dx}{\sqrt{(r^2-x^2)}}\right\}.$$

$$\sigma_\theta = \frac{4\nu}{\pi i}\int_{\bar t}^{t}\varphi'(\zeta)\frac{d\zeta}{\sqrt{[(\zeta-t)(\zeta-\bar t)]}} - \frac{1}{\pi i r^2}\int_{\bar t}^{t}[\varkappa\varphi(\zeta)+(2z-\zeta)\varphi'(\zeta)+\psi(\zeta)]$$

$$\frac{(\zeta-z)\,d\zeta}{\sqrt{(\zeta-t)(\zeta-\bar t)}} + \left\{-\frac{1+2\nu}{4(1-\nu)}U(r,z)\right. \tag{2.5}$$

$$\left.+\frac{c}{4}\int_{L_0}\left[U_1(x,z)\frac{2x^2-r^2}{r^2}+2\,\mathrm{Re}\,W^*(x,z)\frac{x^2-r^2}{r^2}\right]\frac{dx}{\sqrt{(r^2-x^2)}}\right\}.$$

$$\sigma_r = \frac{4(1+\nu)}{\pi i}\int_{\bar t}^{t}\varphi'(\zeta)\frac{d\zeta}{\sqrt{[(\zeta-t)(\zeta-\bar t)]}} - \sigma_z - \sigma_\theta + \left\{-\frac{1+2\nu}{4(1-\nu)}U_0(r,z)\right.$$

$$\left.-\frac{c}{4}\int_{L_0}\left[U_1(x,z)\frac{2x^2-r^2}{r^2}+2\,\mathrm{Re}\,W^*(x,z)\frac{x^2}{r^2}\right]\frac{dx}{\sqrt{(r^2-x^2)}}\right\}.$$

$$\tau_{rz} = -\frac{1}{\pi i r}\int_{\bar t}^{t}[(2z-\zeta)\varphi''(\zeta)+\psi'(\zeta)]\frac{(\zeta-z)\,d\zeta}{\sqrt{[(\zeta-t)(\zeta-\bar t)]}}$$

$$-\left\{\frac{c}{2}\int_{L_0}\mathrm{Im}\,W^*(x,z)\frac{x\,dx}{r\sqrt{(r^2-x^2)}}\right\};$$

$$u = -\frac{1}{2\mu\pi i r}\int_{\bar t}^{t}[\varkappa\varphi(\zeta)+(2z-\zeta)\varphi'(\zeta)+\psi(\zeta)]\frac{(\zeta-z)\,d\zeta}{\sqrt{[(\zeta-t)(\zeta-\bar t)]}}$$

$$-\left\{\frac{c}{4\mu r}\int_{L_0}[\mathrm{Re}\,W^0(x,z)]\frac{x\,dx}{\sqrt{(r^2-x^2)}}\right\} \tag{2.6}$$

$$w = \frac{1}{2\mu\pi i}\int_{\bar t}^{t}[\varkappa\varphi(\zeta)-(2z-\zeta)\varphi'(\zeta)-\psi(\zeta)]\frac{d\zeta}{\sqrt{[(\zeta-t)(\zeta-\bar t)]}}$$

$$-\left\{\frac{c}{4\mu}\int_{L_0}[\mathrm{Im}\,W^0(x,z)]\frac{dx}{\sqrt{(r^2-x^2)}}\right\}.$$

Here $c = (1-2\nu)/(1-\nu)$, $U(r,z)$ will be the body force potentials,

$$\partial U/\partial r = R(r,z), \qquad \partial U/\partial z = Z(r,z)$$

$$W^*(x,z) = W(\zeta,\bar\zeta) = \frac{\partial}{\partial\bar\zeta}\int U^0(\zeta,\bar\zeta)\,d\zeta, \qquad W^0(x,z) = \int U^0(\zeta,\bar\zeta)\,d\zeta, \tag{2.7}$$

$$U_1^0(\zeta,\bar\zeta) = U_1(x,z) = \frac{1}{\pi}\frac{\partial}{\partial x}\int_0^x U(r,z)\frac{r\,dr}{\sqrt{(x^2-r^2)}}.$$

As in the preceding case integration from $\bar t$ to t may be performed along any line lying in the meridional section of the body. As to the integrals shown in braces (and relative to the body forces) we shall have $\int_{L_0}\ldots dx = \int_{-r}^{r}\ldots dx$, that is, integration with respect to x along L_0 denotes integration along a straight line passing within the body, between points $(-r, z)$ and (r, z).

On the axis of symmetry we shall have

$$\sigma_z = 2\varphi'(z) - z\varphi''(z) - \psi'(z), \qquad \tau_{rz} = 0 \tag{2.8}$$

$$\sigma_\theta = \sigma_r = (1+2\nu)\varphi'(z) + \tfrac{1}{2}[z\varphi''(z) + \psi'(z)],$$

$$w = \frac{1}{2\mu}[\varkappa\varphi(z) - z\varphi'(z) - \psi(z)], \qquad u = 0. \tag{2.9}$$

Representations of axisymmetric state components may be derived also by another method.

Let us consider the derivatives of stress and displacement of a plane state with respect to x, involved in (1.11), (1.12) as stress and displacement of a certain elastic antisymmetric plane state as related to the plane yz. By using the Kolosov–Muskhelishvili formulae involving terms reflecting the presence of body forces, and by making corresponding transformations, we shall derive expressions within an accuracy of a constant factor matching the expressions (2.5)–(2.6), if $\int_{L_0} \ldots \mathrm{d}x$ implies here $\int_{-\infty}^{-r} \ldots \mathrm{d}x + \int_r^\infty \ldots \mathrm{d}x$ (that is, integration with respect to x along L_0, as earlier, means integration along a straight line, passing within the body—in the given case from point $(-\infty, z)$ to point $(-r, z)$ and from point (r, z) to point (∞, z).

One can readily see that representations (2.5), (2.6) are of a general character and may also be derived with the aid of a general representation of the solution of the equations of the theory of elasticity in the form of P. F. Papkovich–H. Neuber for the case of axial symmetry. For this purpose the functions involved in P. F. Papkovich's formula should be presented as integrals having the structure of separate components contained in the expressions (2.5)–(2.6) [7].

We shall represent the functions $\varphi(\zeta)$ and $\psi(\zeta)$, involved in the expressions (2.5)–(2.6), by Cauchy-type integrals. Let us change the order of integration, compute the internal integrals and for the case when body forces are absent derive the following representations of stress and transposition [11]

$$\sigma_z = -\frac{1}{2\pi i}\int_L [-2\Phi'(\sigma) + (2z-\sigma)\Phi''(\sigma) + \Psi'(\sigma)]\frac{\mathrm{d}\sigma}{\sqrt{[(\sigma-t)(\sigma-\bar{t})]}},$$

$$\sigma_\theta = \frac{4\nu}{2\pi i}\int_L \Phi'(\sigma)\frac{\mathrm{d}\sigma}{\sqrt{[(\sigma-t)(\sigma-\bar{t})]}} + \frac{c_0}{r} - \frac{1}{2\pi i r^2}$$

$$\times \int_L [\varkappa\Phi(\sigma) + (2z-\sigma)\Phi'(\sigma) + \Psi(\sigma)]\frac{(\sigma-z)\,\mathrm{d}\sigma}{\sqrt{[(\sigma-t)(\sigma-\bar{t})]}}, \tag{2.10}$$

$$\sigma_r = \frac{4(1+\nu)}{2\pi i}\int_L \Phi'(\sigma)\frac{\mathrm{d}\sigma}{\sqrt{[(\sigma-t)(\sigma-\bar{t})]}} - \sigma_z - \sigma_\theta,$$

$$\tau_{rz} = -\frac{1}{2\pi i r}\int_L [(2z-\sigma)\Phi''(\sigma) + \Psi'(\sigma)]\frac{(\sigma-z)\,\mathrm{d}\sigma}{\sqrt{[(\sigma-t)(\sigma-\bar{t})]}};$$

$$w = -\frac{1}{4\pi i\mu}\int_L [-\varkappa\Phi(\sigma) + (2z-\sigma)\Phi'(\sigma) + \Psi(\sigma)]\frac{\mathrm{d}\sigma}{\sqrt{[(\sigma-t)(\sigma-\bar{t})]}},$$

$$u = -\frac{1}{4\pi i\mu r}\int_L [\varkappa\Phi(\sigma) + (2z-\sigma)\Phi'(\sigma) + \Psi(\sigma)]\frac{(\sigma-z)\,\mathrm{d}\sigma}{\sqrt{[(\sigma-t)(\sigma-\bar{t})]}} + \frac{c_0}{r}. \tag{2.11}$$

Here $\sigma = r+iz$ is the fixed point of the contour, functions $\Phi(\sigma)$ and $\Psi(\sigma)$ depict the density of the Cauchy-type integrals, that is, continuous functions of the points of contour L, possessing corresponding derivatives,

$$C_0 = \frac{1}{2\pi i}\int_L [(\varkappa+1)\Phi(\sigma)+\Psi(\sigma)]\,\mathrm{d}\sigma. \tag{2.12}$$

The branch line is to be drawn between points t and $\bar{t}$ within the region.

3. REPRESENTATIONS OF THREE-DIMENSIONAL STATE COMPONENTS OF A HOMOGENEOUS TRANSVERSELY-ISOTROPIC BODY IN TERMS OF ANALYTICAL FUNCTIONS OR CAUCHY-TYPE INTEGRALS

3.1 *Three-dimensional state when heating is involved*

Representations analogous to (2.2)–(2.3) may be derived also for a transversely-isotropic solid of revolution. Let us introduce into the equation (1.1)–(1.2) representations of components in terms of analytical functions $\varphi_1(\zeta_1)$ and $\varphi_2(\zeta_2)$, describing plane strain [15], $\varphi_4(\zeta_4)$, describing the deplanation of a section and $\varphi_3'(\zeta_3)$, describing the stationary temperature field [18]. Here $\zeta_j = z+\lambda_j x$ (when $j = 1, 2, 3, 4$) is the generalized complex variable.

Parameters λ_1, λ_2 are the non-conjugate roots of the characteristic equation

$$\begin{gathered}\left(1-\nu_{xz}^2\frac{E_x}{E_z}\right)\lambda^4+\left[\frac{E_z}{G_{xz}}-2\nu_{xz}(1+\nu_{xy})\right]\lambda^2+(1-\nu_{xy}^2)\frac{E_z}{E_x} = 0,\\ \lambda_3 = i\sqrt{(K_2/K_1)}, \qquad \lambda_4 = i\sqrt{(G_{xz}/G_{xy})}.\end{gathered} \tag{3.1}$$

G_{xy}, G_{xz} will be the moduli of shear, K_1 and K_2 the coefficients of thermal conductivity in the plane of isotropy and along the normal to it.

As a result of transformations one can obtain stress and displacement representations when steady heating is involved. For displacement in case of pure imaginary λ_1 and λ_2 they acquire the form

$$\begin{aligned} w &= \sum_{n=-\infty}^{\infty}\frac{e^{in\theta}}{\pi i}\sum_{j=1}^{3}\int_{\bar{t}_j}^{t_j}P_j\varphi_{jn}(\zeta_j)T_n\left(\frac{2\zeta_j-t_j-\bar{t}_j}{t_j-\bar{t}_j}\right)\frac{\mathrm{d}\zeta_j}{\sqrt{[(\zeta_j-t_j)(\zeta_j-\bar{t}_j)]}},\\ u+iv &= \sum_{n=-\infty}^{\infty}\frac{e^{in\theta}}{\pi}\sum_{j=1}^{4}\int_{\bar{t}_j}^{t_j}f_j\varphi_{jn}(\zeta_j)T_{n+1}\left(\frac{2\zeta_j-t_j-\bar{t}_j}{t_j-\bar{t}_j}\right)\frac{\mathrm{d}\zeta_j}{\sqrt{[(\zeta_j-t_j)(\zeta_j-\bar{t}_j)]}}.\end{aligned} \tag{3.2}$$

Here $\zeta_j = z+\lambda_j x$, $t_j = z+\lambda_j r$; φ_{1n}, φ_{2n}, φ_{4n}, φ_{3n}' are analytical functions in the domains which are in affine correspondence to the region occupied by the axial section of the body and satisfies the conditions

$$\varphi_{jn}(\zeta_j) = (-1)^n\overline{\varphi_{j,-n}(\bar{\zeta})}\,(j = 1, 2, 4), \qquad \varphi_{3n}(\zeta_3) = (-1)^{n-1}\overline{\varphi_{3,-n}(\bar{\zeta}_3)}.$$

P_{1-3} and f_{1-4} are yielded by the formulae

$$
\begin{aligned}
f_{1,2} &= \frac{1+\nu_{xy}}{i\lambda_{1,2}E_z}\left[-\nu_{xz}\lambda_{1,2}^2+(1-\nu_{xy})\frac{E_z}{E_x}\right], \qquad f_4 = 1,\\
f_3 &= \frac{1+\nu_{xy}}{i\lambda_3 E_z}\left[\frac{\alpha_x E_z}{q_0}+(1-\nu_{xy})\frac{E_z}{E_x}-\lambda_3^2\nu_{xz}\right],\\
P_{1,2} &= \frac{1}{E_z}\left[\lambda_{1,2}^2\left(1-\nu_{xz}^2\frac{E_x}{E_z}\right)-\nu_{xz}(1+\nu_{xy})\right],\\
P_3 &= \frac{\alpha_z+\alpha_x\nu_{xz}(E_x/E_z)}{q_0}-\frac{\nu_{xz}}{E_z}(1+\nu_{xy})+\frac{\lambda_3^2}{E_z}\left(1-\nu_{xz}^2\frac{E_x}{E_z}\right),\\
q_0 &= -\frac{\{[\alpha_z+\alpha_x\nu_{xz}(E_x/E_z)]\lambda_3^2+\alpha_x(1+\nu_{xy})\}E_z}{(\lambda_3^2-\lambda_1^2)(\lambda_3^2-\lambda_2^2)[1-\nu_{xz}^2(E_x/E_z)]}.
\end{aligned}
\tag{3.3}
$$

Here α_x and α_z are the coefficients of linear expansion of material in the plane of isotropy and in a direction parallel to the axis of rotation. In this context functions $\varphi_{3n}(\zeta_3)$ are connected with the temperature of the body $T^*(r, z, \theta)$ by the relation

$$
T^* = \sum_{n=-\infty}^{\infty}\frac{e^{in\theta}}{\pi i q_0}\int_{\bar{t}_3}^{t_3}\varphi_{3n}'(\zeta_3)T_n\left(\frac{2\zeta_3-t_3-\bar{t}_3}{t_3-\bar{t}_3}\right)\frac{d\zeta_3}{\sqrt{[(\zeta_3-t_3)(\zeta_3-\bar{t}_3)]}}.
\tag{3.4}
$$

3.2 *Axisymmetric state when heating is involved*

From the expressions (3.2) and the corresponding expressions of stress which are not cited here, or by a method analogous to the one described previously and connected with the application of formulae (1.1), (1.2) as well as with the representations of a plane state in terms of analytical functions, we shall derive representations of axisymmetric state components by contour integrals, containing Cauchy-type integral densities $\Phi(\sigma)$. For the case of pure imaginary non-multiple roots of equations (3.1) we shall find that

$$
T_0^* = \frac{1}{\pi i q_0}\int_{L_3}\frac{\Phi_3'(\sigma_3)\,d\sigma_3}{\sqrt{[(\sigma_3-t_3)(\sigma_3-\bar{t}_3)]}},
\tag{3.5}
$$

$$
\begin{aligned}
\sigma_z &= \frac{1}{\pi i}\sum_{j=1}^{3}\lambda_j^2\int_{L_j}\frac{\Phi_j'(\sigma_j)\,d\sigma_j}{\sqrt{[(\sigma_j-t_j)(\sigma_j-\bar{t}_j)]}},\\
\sigma_r+\sigma_\theta &= \frac{1}{\pi i}\sum_{j=1}^{3}m_j\int_{L_j}\frac{\Phi_j'(\sigma_j)\,d\sigma_j}{\sqrt{[(\sigma_j-t_j)(\sigma_j-\bar{t}_j)]}},\\
\sigma_r-\sigma_\theta &= \frac{1}{\pi i}\sum_{j=1}^{3}(2-m_j)\int_{L_j}\Phi_j'(\sigma_j)\left[2\frac{2\sigma_j-t_j-\bar{t}_j}{(t_j-\bar{t}_j)^2}\right.\\
&\quad\times\left.\left(\frac{2\sigma_j-t_j-\bar{t}_j}{\sqrt{[(\sigma_j-t_j)(\sigma_j-\bar{t}_j)]}}-2\right)-\frac{1}{\sqrt{[(\sigma_j-t_j)(\sigma_j-\bar{t}_j)]}}\right]d\sigma_j,
\end{aligned}
\tag{3.6}
$$

$$\tau_{rz} = -\frac{1}{\pi i}\sum_{j=1}^{3}\lambda_j\frac{\Phi_j'(\sigma_j)}{(t_j-\bar{t}_j)}\left[\frac{2\sigma_j-t_j-\bar{t}_j}{\sqrt{[(\sigma_j-t_j)(\sigma_j-\bar{t}_j)]}}-2\right]\mathrm{d}\sigma_j;$$

$$u = \frac{1}{\pi}\sum_{j=1}^{3}f_j\int_{L_j}\frac{\Phi_j(\sigma_j)}{(t_j-\bar{t}_j)}\left[\frac{2\sigma_j-t_j-\bar{t}_j}{\sqrt{[(\sigma_j-t_j)(\sigma_j-\bar{t}_j)]}}-2\right]\mathrm{d}\sigma_j,$$

$$w = \frac{1}{\pi i}\sum_{j=1}^{3}P_j\int_{L_j}\frac{\Phi_j(\sigma_j)\,\mathrm{d}\sigma_j}{\sqrt{[(\sigma_j-t_j)(\sigma_j-\bar{t}_j)]}}. \tag{3.7}$$

Used here are the notations (3.3) and also

$$m_{1,2} = 1+\nu_{xy}+\lambda_{1,2}^2\nu_{xz}\frac{E_x}{E_z}, \qquad m_3 = 1+\nu_{xy}+\lambda_3^2\nu_{xz}\frac{E_x}{E_z}-\frac{\alpha_x E_x}{q_0}. \tag{3.8}$$

Let us note that the plane and axisymmetric temperature fields $T_1^*(x, z)$ and $T_0^*(r, z)$ may be connected by the operators

$$T_0^* = 2\int_0^r T_1^*\frac{\mathrm{d}x}{\sqrt{(r^2-x^2)}},$$

$$T_0^* = -\frac{1}{\pi}\int_r^x T_1^*\frac{\mathrm{d}x}{\sqrt{(x^2-r^2)}}, \tag{3.9}$$

matching the two different types of superpositions described above. The steady-state temperature T_1^* is placed by operators (3.9) in accordance with the steady-state temperature T_0^*.

If the density of the Cauchy-type integrals Φ_j in formulae (3.5), (3.6) were to be replaced by functions φ_j, analytic in the regions obtained by means of affine transformations $\sigma_j = z+\lambda_j r$ of the region occupied by the axial section of the body, then some components in the right members of (3.6)–(3.8) will be converted to zero, whereas the integrals along the contour may be replaced by doubled integrals from $\bar{t}_j$ to t_j.

By applying superposition as described by the expressions (1.11), (1.12), representations are obtained which correspond to (3.5)–(3.7) within an accuracy of a constant factor.

4. EQUATIONS FOR FUNDAMENTAL BOUNDARY-VALUE PROBLEMS

4.1 *Three-dimensional problems*

In the first fundamental problem the boundary conditions have the form

$$p_z = \sigma_z\cos\alpha+\tau_{rz}\sin\alpha,$$

$$p_r+ip_\theta = \tfrac{1}{2}(\sigma_r+\sigma_\theta)\sin\alpha+\tfrac{1}{2}(\sigma_r-\sigma_\theta+2i\tau_{r\theta})\sin\alpha+(\tau_{rz}+i\tau_{z\theta})\cos\alpha. \tag{4.1}$$

Here p_r, p_z and p_θ are the components of the external loading vector. α is the angle of obliquity of the contour normal of the meridional section to the axis z.

In the representations (2.2) let t and ζ tend to the contour points $t_0 = z_0+ir_0$ and $\sigma = z+ir$, respectively. Be it supposed that the path of integration is the section from $\bar{t}_0$ to t_0 of the contour. Let us introduce (2.2) into (4.1) and, considering that $\cos\alpha = \mathrm{d}r/\mathrm{d}s$, $\sin\alpha = -\mathrm{d}z/\mathrm{d}s$, where $\mathrm{d}s$ is the arc differential of the contour, derive for the case of a homogeneous isotropic body

$$
\begin{aligned}
p_z = & -\sum_{n=-\infty}^{\infty} \frac{e^{in\theta}}{2\pi}\left\{ \frac{1}{r_0^{n+1}} \cdot \frac{d}{ds}\int_{\bar{t}_0}^{t_0} \left[\varphi_n(\sigma) - (2z_0-\sigma)\varphi_n'(\sigma) - \psi_n(\sigma) - \frac{i}{2}\Phi_n(\sigma) \right] \right. \\
& \times T_{n+1}\left(\frac{\sigma - z_0}{r_0 i}\right) \frac{r_0^{n+1}\,dt}{\sqrt{[(\sigma-t_0)(\sigma-\bar{t}_0)]}} + r_0^{n-1}\frac{d}{ds} \\
& \left. \times \int_{\bar{t}_0}^{t_0} \left[\varphi_n(\sigma) - (2z_0-\sigma)\varphi_n'(\sigma) - \psi_n(\sigma) + \frac{i}{2}\Phi_n(\sigma) \right] T_{n-1}\left(\frac{\sigma - z_0}{r_0 i}\right) \frac{r_0^{-n+1}\,d\sigma}{\sqrt{[(\sigma-t_0)(\sigma-\bar{t}_0)]}} \right\}, \\
p_r + ip_\theta = & -\sum_{n=-\infty}^{\infty} \frac{e^{in\theta}}{2\pi i}\left\{ r_0^n \frac{d}{ds}\int_{\bar{t}_0}^{t_0} [(1+4\nu)\varphi_n(\sigma) + (2z_0 - t)\varphi_n'(\sigma) + \psi_n(\sigma)] \right. \\
& \times T_n\left(\frac{\sigma - z_0}{r_0 i}\right) \frac{r_0^{-n}\,d\sigma}{\sqrt{[(\sigma-t_0)(\sigma-\bar{t}_0)]}} + \frac{1}{r_0^{n+2}}\frac{d}{ds} \\
& \times \int_{\bar{t}_0}^{t_0} [(1-4\nu)\varphi_n(\sigma) + (2z_0-\sigma)\varphi_n'(\sigma) + \psi_n(\sigma) + i\Phi_n(\sigma)] \\
& \left. \times T_{n+2}\left(\frac{\sigma - z_0}{r_0 i}\right) \frac{r_0^{n+2}\,d\sigma}{\sqrt{[(\sigma-t_0)(\sigma-\bar{t}_0)]}} \right\}.
\end{aligned}
\tag{4.2}
$$

Be it noted that if the forces on the surface of the body may be presented in the form

$$
\begin{aligned}
p_z &= \sum_{n=-\infty}^{\infty} c_n(z_0, r_0)\, e^{in\theta}, \\
p_r + ip_\theta &= \sum_{n=-\infty}^{\infty} d_n(z_0, r_0)\, e^{in\theta},
\end{aligned}
\tag{4.3}
$$

where

$$
c_{-n}(z_0, r_0) = \overline{c_n(z_0, r_0)}
$$

then by comparing (4.3) and (4.2) we shall derive a system of equations for defining the unknown analytical functions.

The formulae (2.3) make it possible to directly write down the boundary conditions for the case of the second fundamental problem.

4.2 *Axisymmetric problems*

The boundary conditions for the first fundamental problem have the form

$$
p_r = \sigma_r \sin\alpha + \tau_{rz}\cos\alpha, \qquad p_z = \sigma_z \cos\alpha + \tau_{rz}\sin\alpha. \tag{4.4}
$$

Supposing that in (4.2) $n = 0$ and $\Phi_n = 0$, we shall obtain

$$p_z = -\frac{1}{\pi r_0 i}\frac{\mathrm{d}}{\mathrm{d}s}\int_{\bar{t}_0}^{t_0}[\varphi(\sigma)-(2z_0-\sigma)\varphi'(\sigma)-\psi(\sigma)]\frac{(\sigma-z_0)\,\mathrm{d}\sigma}{\sqrt{[(\sigma-t_0)(\sigma-\bar{t}_0)]}},$$

$$\begin{aligned} p_r = &-\frac{1}{2\pi i}\left\{\frac{\mathrm{d}}{\mathrm{d}s}\int_{\bar{t}_0}^{t_0}[(1+4\nu)\varphi(\sigma)+(2z_0-\sigma)\varphi'(\sigma)+\psi(\sigma)]\right. \\ &\times\frac{\mathrm{d}\sigma}{\sqrt{(\sigma-t_0)(\sigma-\bar{t}_0)}}-\frac{1}{r_0^2}\frac{\mathrm{d}}{\mathrm{d}s}\int_{\bar{t}_0}^{t_0}[(1-4\nu)\varphi(\sigma)+(2z_0-\sigma)\varphi'(\sigma)+\psi(\sigma)] \\ &\times\frac{[2(\sigma-z_0)^2+r_0^2]\,\mathrm{d}\sigma}{\sqrt{[(\sigma-t_0)(\sigma-\bar{t}_0)]}}\ . \end{aligned} \tag{4.5}$$

When utilizing the representations in terms of Cauchy-type integrals, the equation for the first fundamental problem may be put down as [11]

$$\begin{aligned} p_z &= \frac{1}{2\pi i r_0}\frac{\mathrm{d}}{\mathrm{d}s}\int_L[-\Phi(\sigma)+(2z_0-\sigma)\Phi'(\sigma)+\Psi(\sigma)]\frac{(\sigma-z_0)\,\mathrm{d}\sigma}{\sqrt{[(\sigma-t_0)(\sigma-\bar{t}_0)]}}, \\ p_r &= \frac{1}{2\pi i r_0^2}\frac{\mathrm{d}}{\mathrm{d}s}\int_L[\Phi(\sigma)+(2z_0-\sigma)\Phi'(\sigma)+\Psi(\sigma)]\left[\sqrt{[(\sigma-t_0)(\sigma-\bar{t}_0)]}+\frac{(\sigma-z_0)^2}{\sqrt{[(\sigma-t_0)(\sigma-\bar{t}_0)]}}\right] \\ &\times\mathrm{d}\sigma-\frac{\sin\alpha}{2\pi i r_0^2}\int_L[(3+4\nu)\Phi(\sigma)+(2z_0-\sigma)\Phi'(\sigma)+\Psi(\sigma)]\frac{(\sigma-z_0)\,\mathrm{d}\sigma}{\sqrt{[(\sigma-t_0)(\sigma-\bar{t}_0)]}}-\frac{c_0}{r^2}\sin\alpha. \end{aligned} \tag{4.6}$$

Let us denote

$$Z_0 = \int_0^s p_z r\,\mathrm{d}s, \qquad R_0 = \int_0^s p_r r^2\,\mathrm{d}s+\int_0^s Z_0\sin\alpha\,\mathrm{d}s, \tag{4.7}$$

here the integration is conducted along the arc, from the point lying on the axis of symmetry. Then equality (4.6) will be transformed into the form

$$\begin{aligned} Z_0 &= \frac{1}{2\pi i}\int_L[-\Phi(\sigma)+(2z_0-\sigma)\Phi'(\sigma)+\Psi(\sigma)]\left[\frac{\sigma-z_0}{\sqrt{[(\sigma-t_0)(\sigma-\bar{t}_0)]}}-1\right]\mathrm{d}\sigma, \\ R_0 &= \frac{1}{2\pi i}\int_L[\Phi(\sigma)+(2z_0-\sigma)\Phi'(\sigma)+\Psi(\sigma)]\left[\sqrt{[(\sigma-t_0)(\sigma-\bar{t}_0)]}-2(\sigma-z_0)\right. \\ &\left.+\frac{(\sigma-z_0)^2}{\sqrt{[(\sigma-t_0)(\sigma-\bar{t}_0)]}}\right]\mathrm{d}\sigma \\ &-\frac{4(1+\nu)}{2\pi i}\int_L\Phi(\sigma)\left\{\int_0^s\left[\frac{\sigma-z_0}{\sqrt{[(\sigma-t_0)(\sigma-\bar{t}_0)]}}-1\right]\sin\alpha\,\mathrm{d}s\right\}\mathrm{d}\sigma. \end{aligned} \tag{4.8}$$

Inasmuch as now the real and imaginary parts of the functions Φ and Ψ are not interrelated, some arbitrariness comes into play.

Let us assume that, as D. I. Sherman has done [16, 17] in reference to a plane problem that

$$\Psi(\zeta) = \frac{K}{2\pi i}\int_L\frac{\overline{\Phi(\sigma)}\,\mathrm{d}\sigma}{\sigma-\zeta}-\frac{1}{2\pi i}\int_L\frac{\bar{\sigma}\Phi'(\sigma)\,\mathrm{d}\sigma}{\sigma-\zeta}. \tag{4.9}$$

where $K = 1$ for the case of the first fundamental problem, $K = -\varkappa$ in the second one. Then from expressions (2.10), (2.11) one can obtain

$$w = \frac{1}{4\mu\pi i}\int_L [\varkappa\Phi(\sigma) - K\overline{\Phi(\sigma)}]\,\mathrm{d}\ln[\sqrt{[(\sigma-t_0)(\sigma-\bar{t}_0)]} + (\sigma - z_0)] - \frac{1}{4\mu\pi i}\int_L \Phi(\sigma)\,\mathrm{d}\left[\frac{\sigma+\bar{\sigma}-2z_0}{\sqrt{[(\sigma-t_0)(\sigma-\bar{t}_0)]}}\right], \tag{4.10}$$

$$u = -\frac{1}{4\pi\mu i r_0}\int_L [\varkappa\Phi(\sigma) + \overline{K\Phi(\sigma)}]\,\mathrm{d}[\sqrt{[(\sigma-t_0)(\sigma-\bar{t}_0)]} - (\sigma - z_0)] - \frac{1}{4\mu\pi i r_0}\int_L \Phi(\sigma)\,\mathrm{d}\left[\frac{(\sigma+\bar{\sigma}-2z_0)(\sigma-z_0)}{\sqrt{[(\sigma-t_0)(\sigma-\bar{t}_0)]}} - (\sigma+\bar{\sigma}-2z_0)\right].$$

Here it is taken into account that c_0 acquires the form

$$c_0 = \frac{1}{2\pi i}\int_L [(\varkappa+1)\Phi(\sigma) + (K-1)\overline{\Phi(\sigma)}]\,\mathrm{d}\sigma. \tag{4.11}$$

The expressions for forces should be written down on the basis of formulae (4.8) as

$$Z_0 = -\frac{1}{2\pi i}\int_L [\Phi(\sigma) - K\overline{\Phi(\sigma)}]\,\mathrm{d}[\sqrt{[(\sigma-t_0)(\sigma-\bar{t}_0)]} - (\sigma - z_0)] + \frac{1}{2\pi i}\int_L \Phi(\sigma)\,\mathrm{d}\left[\frac{(\sigma+\bar{\sigma}-2z_0)(\sigma-z_0)}{\sqrt{[(\sigma-t_0)(\sigma-\bar{t}_0)]}} - (\sigma+\bar{\sigma}-2z_0)\right]$$

$$R_0 = \frac{1}{2\pi i}\int_L [\Phi(\sigma) + K\overline{\Phi(\sigma)}]\,\mathrm{d}[(\sigma-z_0)\sqrt{[(\sigma-t_0)(\sigma-\bar{t}_0)]} - (\sigma-z_0)^2] + \frac{1}{2\pi i}\int_L \Phi(\sigma)\,\mathrm{d}K^*(\sigma, t_0) \tag{4.12}$$

where

$$K^*(\sigma, t_0) = (\sigma+\bar{\sigma}-2z_0)\left[\sqrt{[(\sigma-t_0)(\sigma-\bar{t}_0)]} - 2(\sigma-z_0) + \frac{(\sigma-z_0)^2}{\sqrt{[(\sigma-t_0)(\sigma-\bar{t}_0)]}}\right] - 4(1+\nu)\int_0^s [\sqrt{[(\sigma-t_0)(\sigma-\bar{t}_0)]} - (\sigma-z_0)]\sin\alpha\,\mathrm{d}s.$$

The equalities (4.10) and (4.12) may be regarded as a system of integral equations for the solution of the first and second fundamental problems of the theory of elasticity.

All the preceding formulae are also valid for elastic space, possessing an axisymmetric cavity.

In order to satisfy the equality $u = 0$ when $r = 0$ it is necessary to assume that

$$\frac{1}{2\pi i}\int_L [(\varkappa+1)\Phi(\sigma) + \Psi(\sigma)]\,\mathrm{d}\sigma = c_0 = 0. \tag{4.13}$$

In the formulae (4.7) the beginning of integration becomes the point lying on axis z on top of the cavity.

If formula (4.8) is put down for any point lying below the cavity, we shall obtain

$$Z_0^* = -\frac{1}{\pi i}\int_L \Psi(\sigma)\,d\sigma = \frac{\varkappa+1}{\pi i}\int_L \Phi(\sigma)\,d\sigma \tag{4.14}$$

where $2\pi Z_0^*$ is the resultant of the forces applied to the cavity.

When solving boundary problems it is permissible to use integral equations (4.8) and (4.10). The condition (4.14), considering (4.13) and (4.11), becomes transformed into

$$Z_0^* = \frac{(\varkappa+1)}{\pi i}\int \Phi(\sigma)\,d\sigma = -\frac{K-1}{\pi i}\int_L \overline{\Phi(\sigma)}\,d\sigma. \tag{4.15}$$

This makes it obvious that $2\pi Z_0^*$ corresponds to the action of balanced forces.

By representing the real and imaginary parts of the function by step lines, the height of each step being unknown, one can, with the aid of the derived equations, reduce the numerical solution of the first, second and mixed fundamental problems to the system of linear algebraic equations [11].

5. EXAMPLES FOR THE SOLUTION OF PROBLEMS

5.1 *Axisymmetric problems for an isotropic sphere and for space with a spherical cavity*

(a) For the case when the solid under consideration is restricted by a spherical surface, the relations (4.5) make it possible to a solution of the problem in power series. The values of the unknown analytical functions here may be represented by series of the type

$$\varphi(\sigma) = \sum_{n=-\infty}^{\infty} d_k\sigma^k, \qquad \psi(\sigma) = \sum_{n=-\infty}^{\infty} b_k\sigma^k. \tag{5.1}$$

Now let us substitute these series in the relations (4.5) and utilize formulae

$$\int_{\bar{t}_0}^{t_0} \frac{\sigma^k\,d\sigma}{\sqrt{[(\sigma-t_0)(\sigma-\bar{t}_0)]}} = \pi R^k i P_k(\beta) \tag{5.2}$$

(here $\sigma = R\,e^{i\vartheta}$, $t_0 = R\,e^{i\alpha}$, R is the radius of the sphere: $\beta = \cos\alpha$, $P_k(\beta)$ the Legendre polynomial). Then the integrals involved in the integral relations, will be expressed in terms of Legendre polynomials.

Using the orthogonality property of the Legendre polynomials, we shall derive the formulae for defining all the undetermined coefficients of a_k and b_k. The solutions of the first and second fundamental problems of the theory of elasticity for a sphere and for space with a spherical cavity were demonstrated in this manner. There were presented solutions of the first and second as well as some mixed problems for a hollow sphere.

(b) Representations (2.5), (2.6) can serve as an aid in obtaining a solution in closed form of the first and second fundamental problems of the theory of elasticity for a sphere and for space with a spherical cavity [7]. We shall present here the solution of the first fundamental problem for a sphere.

Let us proceed from (4.5) and take into account that in our case $t_0 = R\,e^{i\alpha}$, $\sigma = R\,e^{i\vartheta}$, $z_0 = R\cos\alpha$, $r_0 = R\sin\alpha$, $ds = R\,d\alpha$, where α and ϑ are the angles measured off in the meridional section of the sphere from axis $z(-\alpha \le \vartheta \le \alpha)$. In the right member of the equation (4.5) let us first integrate by parts and then differentiate in respect to α.

Further we shall introduce the analytical functions $F(\zeta)$ and $F_1(\zeta)$ related to the functions $\varphi(\zeta)$ and $\psi(\zeta)$ by formulae

$$\begin{aligned} \varphi'(\zeta) &= 2\zeta F'(\zeta)+F(\zeta), \\ \psi'(\zeta) &= 2F(\zeta)+\zeta F'(\zeta)-2R^2F''(\zeta)-F_1(\zeta) \end{aligned} \tag{5.3}$$

and reduce the equalities (4.5) to the form

$$\begin{aligned} p_z &= \frac{1}{\pi}\int_{-\alpha}^{\alpha} F_1(\sigma)\frac{e^{3i\vartheta/2}\,d\vartheta}{\sqrt{[2(\cos\vartheta-\cos\alpha)]}}, \\ p_r\sin\alpha &= \frac{1}{\pi}\int_{-\alpha}^{\alpha}[4(1+\nu)\varphi'(\sigma)+4\sigma^2F''(\sigma)-\sigma F_1'(\sigma)-F_1(\sigma)]\sqrt{[2(\cos\vartheta-\cos\alpha)]}\,e^{3i\vartheta/2}\,d\vartheta. \end{aligned} \tag{5.4}$$

Considering the properties of functions $F_1(\sigma)$ and $F(\sigma)$ arising from the properties (2.4) of functions $\varphi(\sigma)$ and $\psi(\sigma)$, we shall obtain

$$p_z = \frac{2}{\pi}\,\mathrm{Re}\int_0^{\alpha} F_1(\sigma)\frac{e^{3i\vartheta/2}\,d\vartheta}{\sqrt{[2(\cos\vartheta-\cos\alpha)]}}. \tag{5.5}$$

Now let us multiply this equation by $\sin\alpha\,d\alpha/\sqrt{[2(\cos\alpha-\cos\gamma)]}$ and integrate it within the limits from 0 to γ. Then we shall change the order of integration in the right member and integrate with respect to α. Next let us differentiate both members with respect to γ, multiply them by $1/2\pi i\,.\,d\tau/(\tau-\zeta)$ and take an integral along the closed contour L. As a result we shall obtain

$$F_1(\zeta) = \frac{1}{2\pi i}\int_L \frac{2R}{\zeta}\frac{e^{-i\gamma/2}\,d\tau}{\tau-\zeta}\frac{d}{d\gamma}\int_0^{\gamma}\frac{p_z\sin\alpha\,d\alpha}{\sqrt{[2(\cos\alpha-\cos\gamma)]}} \tag{5.6}$$

From the equality (4.5) we shall find in an analogous manner that

$$F(\zeta) = \frac{\zeta^{k_1}}{k_1-k_2}\int V(\zeta)\frac{d\zeta}{\zeta^{k_1+1}}-\frac{\zeta^{k_2}}{k_1-k_2}\int V(\zeta)\frac{d\zeta}{\zeta^{k_2+1}}. \tag{5.7}$$

Here k_1 and k_2 are the roots of the equations

$$k^2+(1+2\nu)k+1+\nu = 0$$

$$\begin{aligned} V(\zeta) = \frac{1}{8\pi i}\int_L \Bigg\{& 2\,e^{-3i\gamma/2}\frac{d}{d\gamma}\left[\frac{1}{\sin\gamma}\frac{d}{d\gamma}\int_0^{\gamma}\frac{P_r\sin^2\alpha\,d\alpha}{\sqrt{[2(\cos\alpha-\cos\gamma)]}}\right] \\ & -2i\,e^{-i\gamma}\frac{d}{d\gamma}\left[e^{-i\gamma/2}\frac{d}{d\gamma}\int_0^{\gamma}\frac{p_z\sin\alpha\,da}{\sqrt{[2(\cos\alpha-\cos\gamma)]}}\right]\Bigg\}\frac{d\tau}{\tau-\zeta} \end{aligned} \tag{5.8}$$

The constant integrations which appear when computing undetermined integrals, involved in formula (5.7), should be so selected that the function $F(\zeta)$ would be analytical in the region occupied by the meridional section of the body. For the case of space with a spherical cavity the sign in the right member of the formulae (5.6)–(5.8) is replaced by its opposite sign. The same takes place in front of p_r and p_z.

In the case when displacements are given on the boundary of the sphere, the problem is solved in an analogous manner [7].

For the case when in addition to surface loadings, the sphere or space with a spherical cavity is submitted to the action of body forces, determined by the potential $u(r, z)$, the deduced solutions retain their force, if instead of p_r and p_z we shall introduce p_r^* and p_z^* distinguished by some additions depending on U [8]

$$p_r^* = p_r + \left\{ \frac{\nu}{1-\nu} U(r_0, z_0) + \frac{c}{2} \int_L [U_\beta(x, z_0) + \mathrm{Re}\, W^*(x, z_0)] \frac{x^2\, dx}{r_0^2 \sqrt{(r_0^2 - x^2)}} \right\} \frac{r^0}{R} + \left[\frac{c}{2} \int_L \mathrm{Im}\, W^*(x, z_0) \frac{x\, \mathrm{d}x}{r_0 \sqrt{(r_0^2 - x^2)}} \right] \frac{z_0}{R} \qquad (z_0^2 = R^2 - r_0^2).$$

$$p_z^* = p_z + \left[\frac{1}{2(1-\nu)} U(r_0, z_0) - \frac{c}{2} \int_L \mathrm{Re}\, W^*(x, z_0) \frac{\mathrm{d}x}{\sqrt{r_0^2 - x^2}} \right] \frac{z_0}{R} + \left[\frac{c}{2} \int_L \mathrm{Im}\, W^*(x, z_0) \frac{x\, \mathrm{d}x}{r_0 \sqrt{(r_0^2 - x^2)}} \right] \frac{r_0}{R} \qquad (z_0^2 = R^2 - r_0^2). \tag{5.9}$$

In this case the solution of the second fundamental problem [8] is similar.

5.2 *Axisymmetric problem for an isotropic ellipsoid of revolution and space with an ellipsoidal cavity*

In this case the analytical functions are sought for in the form of such series as

$$\varphi(\zeta) = \sum_{k=0}^{\infty} a_k K_k(\zeta), \qquad \psi(\zeta) = \sum_{k=0}^{\infty} b_k K_k(\zeta). \tag{5.10}$$

For an internal problem $K_k(\zeta) = P_k(\zeta)$ will be the Legendre functions of the first type; for an external problem $K_k(\zeta) = Q_k(\zeta)$ are the Legendre functions of the second type.

Elliptical coordinates $z_0 = \varepsilon\beta$ and $r_0 = \sqrt{(\varepsilon^2 - 1)}\sqrt{(1-\beta^2)}$ are introduced into the plane of the meridional section of the body, wherein

$$\int_{\bar{t}_0}^{t_0} \frac{K_k(\zeta)\, \mathrm{d}\zeta}{\sqrt{[(\zeta - t_0)(\zeta - \bar{t}_0)]}} = \pi i K_k(\varepsilon) P_k(\beta). \tag{5.11}$$

In the procedure analogous to the one applied for solving sphere problems, the coefficients of series (5.10) are in a closed form.

5.3 *Axisymmetric problem for a transversely-isotropic ellipsoid of revolution and space with an ellipsoidal cavity*†

Let the contour of the axial section L be the ellipse

$$z^2/a^2 + r^2/b^2 = 1.$$

Be it assumed that $(-i\lambda_j[b/a]) < 1$. In the equalities (3.6)–(3.7) we shall introduce a substitute $t_j = a\sqrt{(1+\lambda_j^2 b^2/a^2)}\tau_j$ and new parameters $t_{0j} = a\sqrt{(1+\lambda_j^2 b^2/a^2)}\tau_{0j}$. Then we shall obtain representations involving functions $\psi_j(\tau_j)$, analytical in the regions constrained by confocal ellipses, extended along the axis z which correspond to the co-ordinate lines

† The results expounded in 5.3, 5.4 and 3 have been derived jointly with V. S. Volpert.

$\varepsilon = \alpha_j = (1+\lambda_j^2[b^2/a^2])^{-1/2}$ of the elliptical system of co-ordinates

$$z_0 = \varepsilon\beta, \qquad r_0 = \sqrt{(\varepsilon^2-1)(1-\beta^2)}.$$

In this context if t_0 belongs to the contour, then integration in the new representations is conducted along the indicated co-ordinate lines.

The solution of boundary value problems is to be sought in the form

$$\psi_j(\tau_j) = \sum_{k=0}^{\infty} A_{kj} K_k(\tau_j). \tag{5.12}$$

For cases with temperature and forces (or temperatures and displacements), given on the contour, it is possible, taking into account (5.11), to derive coefficients A_{kj} in a closed form.

For example, when temperature and components of displacement are given on the surface of the ellipsoidal cavity, the coefficients A_{3k} are defined from the recurrent relation

$$\begin{aligned} &\frac{k(k+1)}{2k+1} K_{k-1}(\alpha_3) A_{3k} - \frac{(k+2)(k+3)}{2k+5} K_{k+3}(\alpha_3) A_{3,k+2} \\ &\quad = \frac{(2k+3)q_0 a}{2\alpha_3} \int_{-1}^{+1} (\beta^2 - \alpha_3^2) T_0^*(\beta) P_{k+1}(\beta)\, \mathrm{d}\beta. \end{aligned} \tag{5.13}$$

Here

$$A_{30} = -\frac{q_0 a}{2\alpha_3 K_1(\alpha_3)} \int_{-1}^{+1} \beta T_0^*(\beta)\, \mathrm{d}\beta. \tag{5.14}$$

Coefficients A_{1k} and A_{2k} are defined by formulae

$$\begin{aligned} A_{1k} &= \frac{f_2 \mu_{k1} K_k^{(1)}(\alpha_2) - P_2 \mu_{k2} K_k(\alpha_2)}{P_1 f_2 K_k^{(1)}(\alpha_2) K_k(\alpha_1) - P_2 f_1 K_k^{(1)}(\alpha_1) K_k(\alpha_2)}, \qquad (k>0) \\ A_{2k} &= \frac{P_1 \mu_{k2} K_k(\alpha_1) - f_1 \mu_{k1} K_k^{(1)}(\alpha_1)}{P_1 f_2 K_k^{(1)}(\alpha_2) K_k(\alpha_1) - P_2 f_1 K_k^{(1)}(\alpha_1) K_k(\alpha_2)}, \qquad (k>0) \\ A_{10} &= -\frac{\alpha_1\lambda_1}{\alpha_3\lambda_3} \frac{\alpha_3\lambda_3 f_2 \mu_{01} + \alpha_2 P_2 \lambda_2 f_3 Q_0(\alpha_2) A_{03}}{f_1 P_2 \alpha_2 \lambda_2 Q_0(\alpha_2) - f_2 P_1 \alpha_1 \lambda_1 Q_0(\alpha_1)}, \\ A_{20} &= \frac{\alpha_2\lambda_2}{\alpha_3\lambda_3} \frac{\alpha_3\lambda_3 f_1 \mu_{01} + p_1 \alpha_1 \lambda_1 f_3 Q_0(\alpha_1) A_{03}}{f_1 p_2 \alpha_2 \lambda_2 Q_0(\alpha_2) - f_2 p_1 \alpha_1 \lambda_1 Q_0(\alpha_1)} \end{aligned} \tag{5.15}$$

where

$$\begin{aligned} \mu_{k1} &= \frac{2k+1}{2} \int_{-1}^{+1} w_0(\beta) P_k(\beta)\, \mathrm{d}\beta - p_3 A_{3k} K_k(\alpha_3), \\ \mu_{k2} &= -\frac{2k+1}{2} \int_{-1}^{+1} u_0(\beta) P_k^{(1)}(\beta)\, \mathrm{d}\beta - f_3 A_{3k} K_k^{(1)}(\alpha_3). \end{aligned}$$

5.4 *Three-dimensional problem for an isotropic solid of revolution*

(a) In the case when the solid is spherical, the analytical functions may be represented by series type [12]

$$\varphi_n(\zeta) = \sum_{j=0}^{\infty} a_{nj}\zeta^j, \qquad \psi_n(\zeta) = \sum_{j=0}^{\infty} \beta_{nj}\zeta^j, \qquad \Phi_n(\zeta) = \sum_{j=0}^{\infty} \gamma_{nj}\zeta^j. \tag{5.16}$$

Let us substitute these series in (4.2) or in the associated equations for the second fundamental problem, then tend t towards the point of the contour t_0, regard as the path of integration the section $\bar{t}_0 - t_0$ of the contour ($\zeta = \sigma$) and compute the integrals involved here in terms of formulae

$$\frac{1}{\pi i}\int_{\bar{t}_0}^{t_0} t^j T_n\left(\frac{\sigma - z_0}{r_0 i}\right)\frac{d\sigma}{\sqrt{[(\sigma - t_0)(\sigma - \bar{t}_0)]}} = \begin{cases} \dfrac{i^n j!}{(j+n)!}\rho^j P_j^{(n)}(\beta) & \text{for } j \geq n \geq 0 \\ 0 & \text{for } j < n \end{cases} \tag{5.17}$$

Here $\rho = \sqrt{(z^2 + r^2)}$, $\beta = z/\rho$; $P_j^{(n)}(\beta)$ is the added Legendre function of the first type.

By utilizing the orthogonality of the Legendre functions one can define all the coefficients of the series. The number of arbitrary coefficients contained in the solution is sufficient for satisfying the boundary conditions.

(b) For an ellipsoid of revolution or space with an ellipsoidal cavity the solution may be sought in the form

$$\begin{aligned} \varphi_n(\zeta) &= \sum_{j=|n|-1}^{\infty} A_{nj}K_j(\zeta), \\ \psi_n(\zeta) &= \sum_{j=|n|-1}^{\infty} [B_{nj}K_j(\zeta) + (1-2\varepsilon^2)A_{nj}K_{j+1}(\zeta)], \\ \Phi_n(\zeta) &= \sum_{j=|n|-1}^{\infty} C_{nj}K_j(\zeta). \end{aligned} \tag{5.18}$$

(For $n = 0$ summation begins from zero).

Here ε is the elliptical co-ordinate, characterizing the contour of the axial section of the solid.

The coefficients of the series (5.18) for the case of the first and second fundamental problems are in a closed form, if it is considered that

$$\frac{1}{\pi i^{|n|+1}}\int_{\bar{t}_0}^{t_0} \frac{K_j(t)T_n\left(\dfrac{t - z_0}{ir_0}\right)}{\sqrt{[(t-t_0)(t-\bar{t}_0)]}}dt = \begin{cases} \dfrac{(j-n)!}{(j+n)!}K_j^{|n|}(\varepsilon)P_j^{|n|}(\beta) \\ \quad \text{for } j \geq |n| \\ \delta Q_j^{|n|}(\varepsilon)P_j^{-|n|}(\beta) \\ \quad \text{for } j < |n|. \end{cases} \tag{5.19}$$

Here $K_j^{(n)}(\varepsilon)$ will be the added Legendre function of the first $P_j^{(n)}(\varepsilon)$ or second $Q_j^{(n)}(\varepsilon)$ type for the internal and external problems, respectively; $\delta = 0$ for the internal problem, and $\delta = 1$ for the external one. The function $P_j^{-|n|}(\beta)$ is defined by the equation

$$P_j^{-|n|} = \frac{1}{\sin^n\eta}\int_\beta^1 \ldots \int_\beta^1 P_j(\beta)(d\beta)^n \qquad (\cos\eta = \beta).$$

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Абстракт—При помощи суперпозиции устанавливаются зависимости между пространственными состояниями с одной стороны, и элементарными вспомогательными состояниями (плоские состояния и состояния, соответствующие депланации плоских сечений цилиндрических тел), с другой стороны.

В частном случае это зависимости между пространственными осесимметрическими и плоскими состояниями.

Выведенные зависимости используются для решения пространственных задач при помощи методов теории функций комплексного переменного, приложение которых к плоским задачам широко известно.